Supplementary material on enumerating exceptional collections

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February 17, 2015

Abstract

We give more details here on the other Burniat surfaces, Beauville surfaces and the Keum–Naie surface with $K^2 = 4$, including constructions of exceptional collections and answers to Alexeev's question about effective semigroups, giving links to computer code where appropriate.

Section 1 deals with Burniat surfaces. Section 2 is on Beauville surfaces and the final section is on the Keum–Naie surface.

1 Burniat surfaces

1.1 Primary Burniat surfaces with $K^2 = 6$

Exceptional collections on primary Burniat surfaces with $K^2 = 6$ were first constructed and studied in [2] (see also [1]). We apply our own methods here, to give new examples of exceptional collections and to put exceptional collections on the other families of Burniat surfaces into context.

We briefly explain the Burniat line configuration, see [4] for details. Take the three coordinate points P_1, P_2, P_3 in $\mathbb{P}^2$, and label the edges $\overline{A}_0 = P_1P_2$, $\overline{B}_0 = P_2P_3$, $\overline{C}_0 = P_3P_1$. Then let $\overline{A}_1$, $\overline{A}_2$ (respectively $\overline{B}_i$, $\overline{C}_i$) be two general lines passing through P_1 (resp. P_2, P_3). This gives nine lines in total, four passing through each P_i. Blow up the three points P_i to obtain a del Pezzo surface Y of degree 6. The strict transforms of these nine lines (for which we use the same labels) together with the three exceptional curves $\overline{E}_i$, are called the *Burniat line configuration.*

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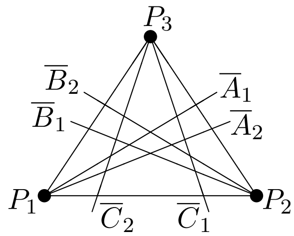

Figure 1: The Primary Burniat configuration with $K^2 = 6$ fig!primary-Bur

The Burniat surface X is a $(\mathbb{Z}/2)^2$-cover of Y branched in the Burniat line configuration, and X is a surface of general type with $p_g = 0$, $K^2 = 6$ and $\mathrm{Tors}(X) = (\mathbb{Z}/2)^6$. The maximal abelian cover A of X is a $(\mathbb{Z}/2)^8$-cover of Y.

The Burniat configuration has four free parameters, and primary Burniat surfaces form a 4-dimensional irreducible connected component of the moduli space of sufaces of general type (see [37]). In particular, $h^1(T_X) = 4$ and $h^2(T_X) = 6$.

In Section 1.5, we show that the primary Burniat surfaces satisfy assumptions (A), exhibiting a basis for $\operatorname{Pic} X/\operatorname{Tors} X$ in terms of reduced pullbacks of irreducible branch divisors. The appendix also lists coordinates for the reduced pullback of each irreducible component of the branch divisor according to Definition 3.4.

We consider the following exceptional collection on Y

$$\Lambda\colon 0,\ e_1,\ e_2,\ e_3,\ e_0,\ 2e_0, \tag{1}$$ eqn!primary-Bur

and use assumption (A) to produce a numerical exceptional collection (L_i) on X. The computer lists acyclic sets $\mathcal{A}(L_i^{-1})$ and $\mathcal{A}(L_j^{-1} \otimes L_i)$, and a systematic search through these enumerates all exceptional collections of numerical type (1).

Theorem 1.1 *There are 81332 exceptional collections $L_0 = \mathcal{O}_X$, $L_1(\tau_1), \dots, L_5(\tau_5)$ on X_6 of numerical type* (1)*. We give a sample of two below.*

	τ_1	τ_2	τ_3	τ_4	τ_5
1	$[1,0,1,0,0,0]$	$[0,0,0,1,0,0]$	$[0,1,0,0,0,1]$	$[0,0,0,0,0,1]$	$[1,1,1,1,0,1]$
2	$[1,0,1,0,0,0]$	$[0,0,0,1,0,0]$	$[0,1,0,0,0,1]$	$[1,1,1,1,0,0]$	$[1,1,1,1,0,1]$

Table 1: Exceptional collections on the primary Burniat surface tab!primary-Bur

Remark 1.1 The precise number of exceptional collections is not important, especially since we have not even taken into account the action of the Weyl group. The basic observation is that there is an abundance of exceptional collections of line bundles on the primary Burniat surface, from which we may choose those with the best properties.

There are 16 exceptional collections on X of numerical type Λ which have no Ext^1-groups, and the two sample exceptional collections are taken from these 16. In all 16 cases, the Ext-groups for the anticanonically extended sequence (12) have the same dimensions, displayed in Table 2.

	0	1	2	3	4	5
0	1	q^2	q^2	q^2	q^2	$6q^2$
1	1	0	0	$2q^2$	$5q^2$	$5q^2$
2	1	0	$2q^2$	$5q^2$	$5q^2$	$6q^2$
3	1	$2q^2$	$5q^2$	$5q^2$	$6q^2$	$6q^2$
4	1	$3q^2$	$3q^2$	$4q^2$	$4q^2$	$4q^2$
5	1	0	q^2	q^2	q^2	$3q^2$

Table 2: Ext-table of an exceptional collection on the primary Burniat surface tab!ext-Burniat

This is the best possible situation, because the product m_2 of any two elements of degree 2 must be identically zero for degree reasons, and all higher products m_n are also zero. Moreover the quasiheight of $\mathbb{E}$ is 4. To summarise, we have:

Proposition 1.1 *Let $\mathbb{E}$ be any one of the 16 exceptional collections on the primary Burniat surface for which there are no* Ext^1*-groups. Then the A_∞-algebra $H^*\mathbb{E}$ is formal, and the product of any two elements of positive degree vanishes. The Hochschild cohomology of each of the corresponding quasi-phantom categories $\mathcal{A}$ is*

$$HH^0(\mathcal{A}) = \mathbb{C},\ HH^1(\mathcal{A}) = 0,\ HH^2(\mathcal{A}) = \mathbb{C}^4,\ HH^3(\mathcal{A}) \supset \mathbb{C}^6.$$

1.2 Secondary Burniat surfaces with $K^2 = 5$

The secondary Burniat surfaces arise when the branch configuration has one or two triple points. We first impose a single triple point P_4 on the three branch lines $\overline{A}_1$, $\overline{B}_1$, and $\overline{C}_2$ (see Figure 2). The $(\mathbb{Z}/2)^2$-cover would then have a $\frac{1}{4}(1,1)$ singularity over P_4, so we blow up at P_4, to obtain a del Pezzo surface Y of degree 5. The induced nonsingular $(\mathbb{Z}/2)^2$-cover X of Y is called a secondary Burniat surface with $K^2 = 5$. Since the cover is unramified over P_4, the torsion group of X is only $(\mathbb{Z}/2)^5$ as opposed to $(\mathbb{Z}/2)^6$ for the primary Burniat surface.

The configuration in Figure 2 has three free parameters, and secondary Burniat surfaces with $K^2 = 5$ form a 3-dimensional irreducible connected component of the moduli space of surfaces of general type (see [5]), so $h^1(T_X) = h^2(T_X) = 3$.

In Appendix 1.6, we give a basis for $\operatorname{Pic} X/\operatorname{Tors} X$ and coordinates for $\operatorname{Pic} X$. In particular, X satisfies assumptions (A). We consider the following exceptional collection of

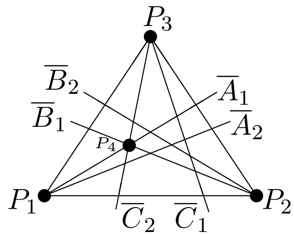

Figure 2: The Secondary Burniat configuration with $K^2 = 5$ fig!secondary-B

line bundles on Y

$$\Lambda\colon 0,\ e_1,\ e_2,\ e_3,\ e_4,\ e_0,\ 2e_0.$$

As usual, we get a numerical exceptional collection (L_i) on X, and we enumerate all exceptional collections on X_5 corresponding to our chosen numerical exceptional collection.

Theorem 1.2 *There are 2597 exceptional collections on X_5 corresponding to $L_0, \dots, L_6$. We give a sample*

$$L_0,\ L_1[0,0,1,1,1],\ L_2[0,1,0,0,1],\ L_3[1,1,1,0,1],\ L_4[0,1,1,0,0],\\ L_5[0,1,1,0,1],\ L_6[1,0,0,1,1],$$

whose Ext-table is found in Table 3.

The sample exceptional collection was chosen because it is the only one of numerical type Λ for which the four line bundles $E_1, \dots, E_4$ corresponding to the (-1)-curves on Y are mutually orthogonal. There are many other exceptional collections of numerical type Λ with very few non-zero Ext^1-groups, but unlike X_6, we do not find any exceptional collections that have no non-zero Ext^1-groups.

Nevertheless, from the table we see that there is no nontrivial composition of two elements of degree 1 in H^*B. Moreover, the elements of degree 1 do not compose with any element below the zigzag.

Proposition 1.2 *The A_∞-algebra of the displayed exceptional collection on the secondary Burniat surface with $K^2 = 5$ is formal, and the product of any two elements of nonzero degree is zero. Moreover, the corresponding quasi-phantom category has Hochschild cohomology*

$$HH^0(\mathcal{A}) = \mathbb{C},\ HH^1(\mathcal{A}) = 0,\ HH^2(\mathcal{A}) = \mathbb{C}^3,\ HH^3(\mathcal{A}) \supset \mathbb{C}^3.$$

1.3 Secondary Burniat surfaces with $K^2 = 4$

We start by describing the secondary Burniat line configurations. Take three coordinate points P_1, P_2, P_3 in $\mathbb{P}^2$, and label the edges $\overline{A}_0 = P_1P_2$, $\overline{B}_0 = P_2P_3$, $\overline{C}_0 = P_3P_1$. Then let $\overline{A}_1$, $\overline{A}_2$ (respectively $\overline{B}_i$, $\overline{C}_i$) be two lines passing through P_1 (resp. P_2, P_3). If these six

	0	1	2	3	4	5	6
0	1	$q+2q^2$	$q+2q^2$	$q+2q^2$	q^2	$3q^2$	$6q^2$
1	1	0	0	0	$2q^2$	$5q^2$	$4q^2$
2	1	0	0	$2q^2$	$5q^2$	$4q^2$	$5q^2$
3	1	0	$2q^2$	$5q^2$	$4q^2$	$5q^2$	$5q^2$
4	1	$2q^2$	$5q^2$	$4q^2$	$5q^2$	$5q^2$	$5q^2$
5	1	$3q^2$	$q+3q^2$	$3q^2$	$3q^2$	$3q^2$	$3q^2$
6	1	q	$q+q^2$	$q+q^2$	$q+q^2$	0	$2q^2$

Table 3: Ext-table of an exceptional collection on the secondary Burniat surface with $K^2 = 5$ tab!ext-Burniat

lines are chosen to be general, then we have the *Burniat line configuration.* This gives nine lines in total, four passing through each P_i.

The secondary Burniat line configurations arise when the above line configuration is allowed to have one or two triple points. We consider the case of two triple points, which we denote P_4 and P_5. There are two possibilities, leading to two different secondary Burniat surfaces with $K_X^2 = 4$ (see Figure 3). If P_4 and P_5 do not lie on the same branch line, then the blow up Y of $\mathbb{P}^2$ at $P_1, \ldots, P_5$ is a del Pezzo surface of degree 4, and the $(\mathbb{Z}/2)^2$-cover is called *non-nodal.* If P_4 and P_5 do lie on a single branch line (in Fig. 3, this line is $\overline{A}_1$), then Y is a weak del Pezzo surface, and the $(\mathbb{Z}/2)^2$-cover is called *nodal* because the canonical model of X has a $\frac{1}{2}(1,1)$ singularity. In both cases, the second triple point causes the torsion group of X to drop to $(\mathbb{Z}/2)^4$.

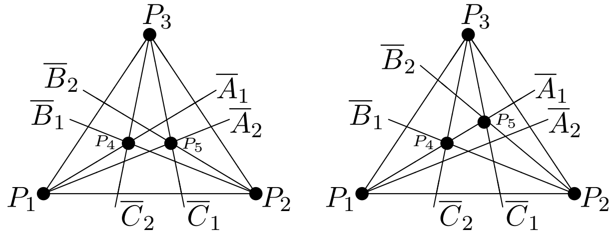

Figure 3: The secondary Burniat configurations with $K^2 = 4$ (nodal configuration is on the right) fig!secondary-B

Both configurations in Figure 3 have two free parameters, so that we obtain two 2-dimensional families of secondary Burniat surfaces with $K^2 = 4$. We recall some facts from [5] and [6]. The non-nodal case again forms an irreducible connected component of the moduli space, with $h^1(T_X) = 2$ and $h^2(T_X) = 0$. The nodal case is more interesting: the 2-dimensional family is a proper subset of a 3-dimensional irreducible connected component of the moduli space. In fact, $h^1(T_{X^n}) = 3$ and $h^2(T_{X^n}) = 1$, and there is a 3-dimensional family of *extended Burniat surfaces* (see [5]), each of which is a $(\mathbb{Z}/2)^2$-cover of a generalisation of the nodal Burniat configuration. We do not directly consider extended Burniat surfaces in this article.

The data listed in Appendix 1.7 shows that both X_4 and X_4^n satisfy assumption (A). Choosing the numerical exceptional collection

$$\Lambda\colon 0,\ e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_0,\ 2e_0,$$

we enumerate all exceptional collections on X_4^n corresponding to Λ.

Theorem 1.3 *There are 13 exceptional collections on X_4^n of numerical type Λ. Here is a sample exceptional collection*

$$\begin{aligned} L_0,\ L_1[1,0,1,0],\ L_2[0,1,0,1],\ L_3[0,0,1,1],\ L_4[0,1,1,0],& \\ L_5,\ L_6[0,1,0,1],\ L_7[1,0,1,1],& \quad (2) \end{aligned}$$

eqn!secondary-B

whose Ext*-table is found in Table 4.*

	0	1	2	3	4	5	6	7
0	1	q^2	q^2	q^2	q^2	q^2	$3q^2$	$6q^2$
1	1	$q+q^2$	0	$q+q^2$	$q+q^2$	$2q^2$	$5q^2$	$3q^2$
2	1	0	0	0	$2q^2$	$5q^2$	$3q^2$	$4q^2$
3	1	0	$q+q^2$	$2q^2$	$5q^2$	$3q^2$	$4q^2$	$4q^2$
4	1	0	$2q^2$	$5q^2$	$3q^2$	$4q^2$	$4q^2$	$4q^2$
5	1	$2q^2$	$5q^2$	$3q^2$	$4q^2$	$4q^2$	$4q^2$	$4q^2$
6	1	$3q^2$	$q+2q^2$	$2q^2$	$2q^2$	$2q^2$	$2q^2$	$2q^2$
7	1	$2q$	q	q	q	q	$2q+q^2$	q^2

Table 4: Ext-table of an exceptional collection on the nodal Burniat surface with $K^2 = 4$ tab!ext-Burniat

The non-nodal surface X_4 has six exceptional collections of numerical type Λ, but it is difficult to find one for which the A_∞-algebra is obviously formal, because there are too many nonzero Ext^1-groups. We use the Weyl group action on the del Pezzo surface Y to obtain the following numerical exceptional collection

$$\Lambda'\colon 0,\ e_4,\ e_2,\ e_5,\ e_1,\ e_3,\ e_0,\ 2e_0.$$

This is just a permutation of the order in which we blow up the points in $\mathbb{P}^2$ to construct Y.

Theorem 1.4 *There are 40 exceptional collections on X_4 of numerical type Λ', and we exhibit one with the minimum number of* Ext^1*-groups*

$$L_0,\ L_1[0,0,0,1],\ L_2[0,1,1,0],\ L_3[1,0,1,0],\ L_4[0,1,0,1],\ L_5[1,0,0,0],\ L_6[1,1,1,1],\ L_7[1,1,1,0]. \quad (3)$$

eqn!secondary-B

The Ext*-table of* (3) *is displayed in Table 5.*

	0	1	2	3	4	5	6	7
0	1	q^2	q^2	q^2	q^2	q^2	$3q^2$	$6q^2$
1	1	0	$q+q^2$	0	$q+q^2$	$2q^2$	$5q^2$	$3q^2$
2	1	$q+q^2$	$q+q^2$	$q+q^2$	$2q^2$	$5q^2$	$3q^2$	$4q^2$
3	1	0	0	$2q^2$	$5q^2$	$3q^2$	$4q^2$	$4q^2$
4	1	0	$2q^2$	$5q^2$	$3q^2$	$4q^2$	$4q^2$	$4q^2$
5	1	$2q^2$	$5q^2$	$3q^2$	$4q^2$	$4q^2$	$4q^2$	$4q^2$
6	1	$3q^2$	q^2	$2q^2$	$2q^2$	$2q^2$	$2q^2$	$2q^2$
7	1	$2q$	q	q	q	q	q	q^2

Table 5: Ext-table of an exceptional collection on the non-nodal Burniat surface with $K^2 = 4$

tab!ext-Burniat

We see that both the non-nodal and nodal secondary Burniat surfaces with $K^2 = 4$ have quite a few nonzero Ext^1-groups, since we do not have as much freedom to search for "good" exceptional collections. Nevertheless, a careful examination of the tables shows that no two elements of degree 1 are composable. Thus in both cases, the A_∞-algebra is formal, and the height is 4.

We summarise our results on Burniat surfaces with $K^2 = 6, 5, 4$.

Theorem 1.5 *Every primary or secondary Burniat surface has at least one exceptional collection of maximal length whose A_∞-algebra is formal. Moreover, the product of any two elements of positive degree vanishes, and the height is* 4*. Thus the Hochschild cohomology of each corresponding quasiphantom category is*

$$HH^0(\mathcal{A}) = H^0(\mathcal{O}_X),\ HH^1(\mathcal{A}) = 0,\ HH^2(\mathcal{A}) = H^1(T_X),\ HH^3(\mathcal{A}) \supset H^2(T_X).$$

1.4 Tertiary Burniat surface with $K^2 = 3$

Imposing a third triple point on the branch configuration (see Fig. 4) gives a tertiary Burniat surface X_3 with $K_X^2 = 3$. The weak del Pezzo surface Y has three (-2)-curves,

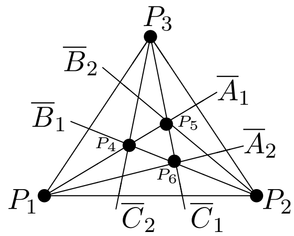

Figure 4: The tertiary Burniat configuration with $K^2 = 3$ fig!tertiary-Bu

$\overline{A}_1$, $\overline{B}_1$ and $\overline{C}_1$, and the canonical model of X_3 is a $(\mathbb{Z}/2)^2$-cover of a 3-nodal cubic. The torsion group of X_3 is $(\mathbb{Z}/2)^3$.

Here the moduli space gets quite involved, and we follow the description of [6]. The tertiary Burniat surfaces form a 1-dimensional irreducible family, inside a 4-dimensional irreducible component of the moduli space. The *extended tertiary Burniat surfaces* form an open subset of this irreducible component, and the remainder consists of $(\mathbb{Z}/2)^2$-covers of certain singular cubic surfaces. Our main point of interest is that $h^1(T_X) = 4$, and $h^2(T_X) = 0$.

In Appendix 1.8, we show that X satisfies assumption (A). We use the computer to enumerate all exceptional collections on X of numerical type

$$\Lambda\colon 0,\ e_1,\ e_2,\ e_3,\ e_4,\ e_5,\ e_6,\ e_0,\ 2e_0.$$

Lemma 1.1 *There are no exceptional collections of line bundles of numerical type Λ on the tertiary Burniat surface.* □

Remark 1.2 Our systematic search does yield exceptional collections $\mathbb{E}'$ of length seven with numerical type $0, e_1, \ldots, e_6$, but in each case, there are no line bundles corresponding to e_0 or $2e_0$ which extend $\mathbb{E}'$. We have also checked part of the orbit of Λ under the action of the Weyl group of Y, and although we find some exceptional collections on X_3 of length eight, we do not find any of length nine.

1.4.1 The E_6-symmetry

In order to find an exceptional collection of line bundles on X_3, we choose a different numerical exceptional collection Λ_1, using the E_6-symmetry of $\operatorname{Pic} Y$ and the Borel–de Siebenthal procedure. As an example, we consider the sublattice $3A_2$ inside the extended Dynkin diagram $\widetilde{E}_6$, which corresponds to a singular del Pezzo surface Y' with $3 \times \frac{1}{3}(1,2)$ singularities. The minimal resolution $\widetilde{Y}$ is a toric surface with a cycle of nine rational curves with self-intersections

$$-(-2)-(-1)-(-2)-(-2)-(-1)-(-2)-(-2)-(-1)-(-2)-$$

To construct $\widetilde{Y}$, choose points P_1, P_2, P_3 in general position in $\mathbb{P}^2$. Blow up each P_i once, and blow up the infinitely near points Q_1, Q_2, Q_3, where Q_i is supported at P_i with tangent direction P_iP_{i+1}. Alternatively, $\widetilde{Y}$ is the minimal resolution of the quotient of $\mathbb{P}^2$ by the $\mathbb{Z}/3$-action $\frac{1}{3}(0,1,2)$, which has three fixed points. We fix a geometric marking on $\widetilde{Y}$ so that the strict transform of the exceptional curve over P_i has class $e_i - e_{3+i}$ in $\operatorname{Pic}\widetilde{Y}$, and the exceptional curve over Q_i has class e_{3+i}. Then the cycle of curves described above have numerical classes

$$e_1 - e_4,\ e_4,\ e_0 - e_1 - e_2 - e_4,\ e_2 - e_5,\ e_5,\ e_0 - e_2 - e_3 - e_5,\ e_3 - e_6,\ e_6,\ e_0 - e_1 - e_3 - e_6$$

Taking cumulative sums of these classes gives a numerical exceptional collection Λ_1 on any del Pezzo surface of degree three:

$$\Lambda_1\colon 0,\ e_1 - e_4,\ e_1,\ e_0 - e_2 - e_4,\ e_0 - e_4 - e_5,\ e_0 - e_4, \\ 2e_0 - e_2 - e_3 - e_4 - e_5,\ 2e_0 - e_2 - e_4 - e_5 - e_6,\ 2e_0 - e_2 - e_4 - e_5. \quad (4)$$ eqn!Lambda1

We again search for exceptional collections of type Λ_1 on X, and again we do not find any. Fortunately, this time we do find exceptional collections on X using the Weyl group action on Y.

Theorem 1.6 *There are exceptional collections on X_3 corresponding to certain numerical exceptional collections in the Weyl group orbit of Λ_1. For example, let Λ_1' be the numerical exceptional collection obtained from Λ_1 by swapping e_1 and e_2. Then*

$$L_0,\ L_1[0,0,1],\ L_2[0,1,1],\ L_3[1,0,0],\ L_4[0,1,1],\ L_5[0,0,1], \\ L_6[1,1,0],\ L_7[0,0,0],\ L_8[0,1,0] \quad (5)$$ eqn!tertiary-Bu

is an exceptional collection on X_3 of numerical type Λ_1', whose Ext*-table is found in Table 6.*

Remark 1.3 We have not studied the whole orbit of numerical exceptional collections, because the Weyl group is quite large, but we can give an overview based on probabilistic methods. It seems that approximately two thirds of the orbit of Λ_1 do not give any exceptional collections on X, and the remaining numerical types typically correspond to anywhere between one and 21 exceptional collections on X. We see that exceptional collections are much more scarce on X_3 than for the other Burniat surfaces.

1.4.2 The A_∞-algebra

The exceptional collection (5) was chosen to have the fewest nonzero Ext^1-groups, but there are six of them. Of these, there are no three above the zigzag that may be composed with one another under m_3. Thus m_3 is identically zero on H^*B for degree reasons, and the A_∞-algebra of $\mathbb{E}$ is formal. There is a single possible product of two elements of degree 1, coming from the chain $E_1 \to E_4 \to E_7$. It is not clear whether this product is zero.

	0	1	2	3	4	5	6	7	8
0	1	0	q^2	q^2	q^2	$2q^2$	$2q^2$	$2q^2$	$3q^2$
1	1	q^2	q^2	$q+2q^2$	$2q^2$	$2q^2$	$2q^2$	$3q^2$	$3q^2$
2	1	0	0	q^2	$q+2q^2$	q^2	$2q^2$	$2q^2$	$2q^2$
3	1	0	q^2	q^2	q^2	$2q^2$	$2q^2$	$2q^2$	$3q^2$
4	1	q^2	q^2	$q+2q^2$	$2q^2$	$2q^2$	$2q^2$	$3q^2$	$3q^2$
5	1	0	0	q^2	$q+2q^2$	q^2	$2q^2$	$2q^2$	$2q^2$
6	1	0	q^2	q^2	q^2	$2q^2$	$2q^2$	$2q^2$	$3q^2$
7	1	q^2	q^2	$q+2q^2$	$2q^2$	$2q^2$	$2q^2$	$3q^2$	$3q^2$
8	1	0	0	q^2	$q+2q^2$	q^2	$2q^2$	$2q^2$	$2q^2$

Table 6: Ext-table of an exceptional collection on the tertiary Burniat surface with $K^2 = 3$ tab!ext-Burniat

To compute the Hochschild cohomology, we first consider the pseudoheight of $\mathbb{E}$. Examining the table, we see that the pseudoheight is 3, because

$$e(E_1, E_4) + e(E_4, E_7) + e(E_7, E_1 \otimes \omega_X^{-1}) + 2 - 2 = 3.$$

In fact, this cycle of line bundles is the only one contributing 3 to the pseudoheight. In other words, the first page of the spectral sequence converging to $NHH^\bullet(\mathbb{E}, X)$ has a single term of total degree 3:

$$\mathrm{Ext}^1(E_1, E_4) \otimes \mathrm{Ext}^1(E_4, E_7) \otimes \mathrm{Ext}^3(E_7, S^{-1}(E_1)) \subset \mathbf{E}^1_{-2,5}.$$

The differential d_1 on the first page maps this term to the direct sum of the following three spaces

$$\begin{gathered} \mathrm{Ext}^2(E_1, E_7) \otimes \mathrm{Ext}^3(E_7, S^{-1}(E_1)) \\ \mathrm{Ext}^1(E_1, E_4) \otimes \mathrm{Ext}^4(E_4, S^{-1}(E_1)) \\ \mathrm{Ext}^1(E_4, E_7) \otimes \mathrm{Ext}^4(E_7, S^{-1}(E_4)) \end{gathered}$$

in $\mathbf{E}^1_{-1,5}$.

If we can show that any of the maps are nonzero, then it follows that $\mathbf{E}^2_{-2,5} = 0$ and thus $h(\mathbb{E}) = 4$. We do not currently know of a practical method for computing nontrivial products in H^*B, but the following rough idea should work.

Observe that $\mathrm{Ext}^k(E_i, E_j) = H^k(E_i^{-1} \otimes E_j)$. We write $L_{ij} = E_i^{-1} \otimes E_j$, and so we are actually checking injectivity of the cup product $H^1(L_{14}) \otimes H^1(L_{47}) \xrightarrow{\cup_X} H^2(L_{17})$. It is difficult to compute $\cup_X$ explicitly on X, so we pushforward each L_{ij} to Y, and compare with the cup product $\cup_Y$ on Y. We have

$$H^1(\varphi_* L_{14}) \otimes H^1(\varphi_* L_{47}) \xrightarrow{\cup_Y} H^2(\varphi_* L_{14} \otimes \varphi_* L_{47}) \xrightarrow{\mu} H^2(\varphi_* L_{17}), \qquad (6)$$

eqn!cupY

where μ is induced by the natural map $\varphi_* L_{14} \otimes \varphi_* L_{47} \to \varphi_*(L_{14} \otimes L_{47})$. By comparing the definition of $\cup_Y$ using Čech complexes with that of $\cup_X$, we see that the composite map displayed in (6) is equal to $\cup_X$.

It remains to compute the cup product on Y, which can be done by chasing exact sequences, and to check that μ is injective. We hope to finish this in the near future.

1.5 Data for Primary Burniat surface

primary-Burniat

The maps determining the covers $A \to X \to Y$ are $\Phi\colon H_1(Y - \Delta, \mathbb{Z}) \to G$ and $\Psi\colon H_1(Y - \Delta, \mathbb{Z}) \to G \oplus T$. We tabulate them below.

Γ	$\overline{A}_0$	$\overline{A}_1$	$\overline{A}_2$	$\overline{B}_0$	$\overline{B}_1$	$\overline{B}_2$	$\overline{C}_0$	$\overline{C}_1$	$\overline{C}_2$
$\Phi(\Gamma)$	g_1	g_1	g_1	g_2	g_2	g_2	$g_1 + g_2$	$g_1 + g_2$	$g_1 + g_2$
$\Psi(\Gamma) - \Phi(\Gamma)$	0	g_3	g_4	0	g_5	g_6	g_7	g_8	$\sum_{i=3}^{8} g_i$

The images of the exceptional curves are obtained in the usual way from equation (1) of Sec. 2,

$$\Phi(\overline{E}_1) = g_2,\ \Phi(\overline{E}_2) = g_1 + g_2,\ \Phi(\overline{E}_3) = g_1,\ \text{etc.}$$

The following reduced pullbacks are a basis for $\operatorname{Pic} X/\operatorname{Tors} X$,

$$e_0 = C_0 + E_1 + E_3,\ e_1 = E_1,\ e_2 = E_2,\ e_3 = E_3.$$

According to these generators, the coordinates on $\operatorname{Pic} X$ are

	Multidegree				Torsion
$\mathcal{O}_X(A_0)$	1	-1	-1	0	$[1,1,0,0,0,1]$
$\mathcal{O}_X(A_1)$	1	-1	0	0	$[1,0,0,0,1,0]$
$\mathcal{O}_X(A_2)$	1	-1	0	0	$[0,1,0,0,1,0]$
$\mathcal{O}_X(B_0)$	1	0	-1	-1	$[0,0,1,1,0,1]$
$\mathcal{O}_X(B_1)$	1	0	-1	0	$[0,0,1,0,1,0]$
$\mathcal{O}_X(B_2)$	1	0	-1	0	$[0,0,0,1,1,0]$
$\mathcal{O}_X(C_0)$	1	-1	0	-1	0
$\mathcal{O}_X(C_1)$	1	0	0	-1	$[0,0,0,0,1,1]$
$\mathcal{O}_X(C_2)$	1	0	0	-1	$[0,0,0,0,1,0]$

and $\mathcal{O}_X(K_X) = \mathcal{O}_X(3,-1,-1,-1)[0,0,0,0,0,1]$ by equation (8).

ndary-Burniat-5

1.6 Data for Secondary Burniat surface with $K^2 = 5$

The map $\Phi\colon H_1(\mathbb{P}^2 - \Delta, \mathbb{Z}) \to (\mathbb{Z}/2)^2$ is the same as for the primary Burniat surface, but the triple point at P_4 changes Ψ. Indeed, we have $\Psi_5(\overline{A}_1 + \overline{B}_1 + \overline{C}_2) = 0$, which kills one factor of the torsion group. Thus the maximal abelian cover $\psi_5\colon A \to Y$ is determined by

Γ	$\overline{A}_0$	$\overline{A}_1$	$\overline{A}_2$	$\overline{B}_0$	$\overline{B}_1$	$\overline{B}_2$	$\overline{C}_0$	$\overline{C}_1$	$\overline{C}_2$
$\Psi_5(\Gamma) - \Phi(\Gamma)$	0	g_3	g_4	0	g_5	g_6	g_7	$g_4 + g_6 + g_7$	$g_3 + g_5$

and the torsion group is generated by $g_3^*, \dots, g_7^*$.

The following reduced pullbacks are a basis for the free part of $\operatorname{Pic} X$

$$e_0 = C_0 + E_1 + E_3,\ e_1 = E_1,\ e_2 = E_2,\ e_3 = E_3,\ e_4 = C_0 - C_2 + E_1,$$

with intersection form $\operatorname{diag}(1, -1, -1, -1, -1)$. Note that $\overline{E}_4$ is not a branch divisor, which explains the funny choice for e_4.

The coordinates of $\operatorname{Pic} X$ according to this basis are:

	Multidegree					Torsion
$\mathcal{O}_X(A_0)$	1	-1	-1	0	0	$[1,1,0,0,0]$
$\mathcal{O}_X(A_1)$	1	-1	0	0	-1	$[1,0,0,0,0]$
$\mathcal{O}_X(A_2)$	1	-1	0	0	0	$[0,1,0,0,1]$
$\mathcal{O}_X(B_0)$	1	0	-1	-1	0	$[0,0,1,1,0]$
$\mathcal{O}_X(B_1)$	1	0	-1	0	-1	$[0,0,1,0,0]$
$\mathcal{O}_X(B_2)$	1	0	-1	0	0	$[0,0,0,1,1]$
$\mathcal{O}_X(C_0)$	1	-1	0	-1	0	0
$\mathcal{O}_X(C_1)$	1	0	0	-1	0	$[0,0,0,0,1]$
$\mathcal{O}_X(C_2)$	1	0	0	-1	-1	0

Thus $\mathcal{O}_X(K_X) = \mathcal{O}_X(3, -1, -1, -1, -1)[0, 0, 0, 0, 1]$.

ndary-Burniat-4

1.7 Data for Secondary Burniat surfaces with $K^2 = 4$

tiary-Burniat-3

1.8 Data for Tertiary Burniat surfaces with $K^2 = 3$

The maps determining the covers $A \to X \to Y$ are $\Phi\colon H_1(Y - \Delta, \mathbb{Z}) \to (\mathbb{Z}/2)^2$ and $\Psi\colon H_1(Y - \Delta, \mathbb{Z}) \to (\mathbb{Z}/2)^5$. We tabulate them below.

Γ	$\overline{A}_0$	$\overline{A}_1$	$\overline{A}_2$	$\overline{B}_0$	$\overline{B}_1$	$\overline{B}_2$	$\overline{C}_0$	$\overline{C}_1$	$\overline{C}_2$
$\Phi(\Gamma)$	g_1	g_1	g_1	g_2	g_2	g_2	$g_1 + g_2$	$g_1 + g_2$	$g_1 + g_2$
$\Psi(\Gamma) - \Phi(\Gamma)$	0	g_3	g_4	0	g_5	$g_3 + g_4 + g_5$	$g_3 + g_4$	$g_4 + g_5$	$g_3 + g_5$

The images of the exceptional curves are obtained in the usual way from equation (1) of Sec. 2.

The following reduced pullbacks form a basis of the free part of $\mathrm{Pic}(X_3)$:

$$e_0 = C_0 + E_1 + E_3,\ e_1 = E_1,\ e_2 = E_2,\ e_3 = E_3,$$
$$e_4 = C_0 - C_2 + E_1,\ e_5 = B_0 - B_2 + E_3,\ e_6 = A_0 - A_2 + E_2.$$

We note that $\varphi^*(\overline{E}_i) = 2e_i$ for $i = 4, 5, 6$. The coordinates for the reduced pullback of each irreducible component of the branch divisor are

	Multidegree							Torsion
$\mathcal{O}_X(A_0)$	1	-1	-1	0	0	0	0	$[1,1,0]$
$\mathcal{O}_X(A_1)$	1	-1	0	0	-1	-1	0	$[1,0,1]$
$\mathcal{O}_X(A_2)$	1	-1	0	0	0	0	-1	$[1,1,0]$
$\mathcal{O}_X(B_0)$	1	0	-1	-1	0	0	0	$[0,0,1]$
$\mathcal{O}_X(B_1)$	1	0	-1	0	-1	0	-1	$[1,0,1]$
$\mathcal{O}_X(B_2)$	1	0	-1	0	0	-1	0	$[0,0,1]$
$\mathcal{O}_X(C_0)$	1	-1	0	-1	0	0	0	0
$\mathcal{O}_X(C_1)$	1	0	0	-1	0	-1	-1	$[1,0,1]$
$\mathcal{O}_X(C_2)$	1	0	0	-1	-1	0	0	0

Thus $\mathcal{O}_X(K_X) = \mathcal{O}_X(3,-1,-1,-1,-1,-1,-1)[1,0,1]$ by equation (8).

1.9 The Nef cone of Burniat 3

On the del Pezzo surface $f_i = e_0 - e_i$, $h_0 = e_0$, $2e_0 - \sum e_i + e_j + e_k$ for $(j,k) = (5,6),(4,6),(4,5),(3,4),(2,5),(1,6)$ Write these in terms of Δ $A_0 + C_2$, $C_0 + B_2$, $B_0 + A_2$, $A_2 + B_2$, $A_2 + C_2$, $B_2 + C_2$

These are all the Cremona hyperplanes except for the three effective -2-curves. $2e_0 - \sum_{i \in S} e_i$ for $S = (1,2,3),(1,2,4),(1,2,5),(1,2,6),(1,3,4),(1,3,5),(1,3,6),(1,4,6),(1,5,6),(2,3,4),(2,3,5),(2$ $(2,4,5),(2,5,6),(3,4,5),(3,4,6),(4,5,6)$ In terms of f_i etc

$2h_0 - E_1 - E_2 - E_3 = f_1 + B_0 = f_2 + C_0 = f_3 + A_0$,

$f_4 + A_0$, $f_4 + C_0 = f_1 + C_2$, $f_4 + B_0 = f_2 + C_2$, $f_4 + B_2$, $f_4 + A_2$,

$f_5 + A_0 = f_1 + B_2$, $f_5 + C_0$, $B_0 + f_5$, $f_5 + A_2$,$C_2 + f_5$,

$f_6 + A_0$, $f_6 + C_0$, $B_0 + f_6$, $f_6 + B_2$, $C_2 + f_6$,

$h' = 2h_0 - E_4 - E_5 - E_6$ (not written in terms of fibr.)

$K = (3 - 1 - 1 - 1 - 1 - 1 - 1)$

These are ones that can not be written as $2 - ijk$ + a -2-curve but can be written as $2e_0 - ijkl$ + branch. (3 -2 -1 -1 -1 0 -1), (3 -2 -1 -1 0 -1 -1),

(3 -1 -2 -1 -1 -1 0), (3 -1 -2 -1 0 -1 -1),

(3 -1 -1 -2 -1 -1 0), (3 -1 -1 -2 -1 0 -1),

(3 -2 -1 0 -1 0 -1), (3 -2 0 -1 0 -1 -1), (3 -1 -2 0 -1 -1 0), (3 -1 0 -2 -1 -1 0), (3 -1 0 0 -1 -1 -2), (3 0 -2 -1 0 -1 -1), (3 0 -1 -2 -1 0 -1), (3 0 -1 0 -1 -2 -1), (3 0 0 -1 -2 -1 -1),

$K + h$ (4 -2 -2 -2 -1 -1 -1),

(5 -3 -2 -2 -1 -1 -2), (5 -2 -3 -2 -1 -2 -1), (5 -2 -2 -3 -2 -1 -1)

2 Beauville surfaces

app!Beauville

In this appendix we apply our methods to two Beauville surfaces. Each is an abelian cover of $\mathbb{P}^1 \times \mathbb{P}^1$ satisfying assumptions (A). Thus we may write any line bundle on X as $\mathcal{O}_X(a,b)(\tau)$. We recall some facts about numerical exceptional collections on such abelian covers of $\mathbb{P}^1 \times \mathbb{P}^1$ from [24].

lem!GS **Lemma 2.1** *1. A sequence $\mathcal{O}, L_1, L_2, L_3$ of line bundles on X is numerically exceptional if and only if it belongs to one of the four numerical types:*

$$\begin{aligned}
(I_c)&\ \mathcal{O},\ \mathcal{O}(-1,0),\ \mathcal{O}(c-1,-1),\ \mathcal{O}(c-2,-1),\\
(II_c)&\ \mathcal{O},\ \mathcal{O}(0,-1),\ \mathcal{O}(-1,c-1),\ \mathcal{O}(-1,c-2),\\
(III_c)&\ \mathcal{O},\ \mathcal{O}(-1,c),\ \mathcal{O}(-1,c-1),\ \mathcal{O}(-2,-1),\\
(IV_c)&\ \mathcal{O},\ \mathcal{O}(c,-1),\ \mathcal{O}(c-1,-1),\ \mathcal{O}(-1,-2),
\end{aligned}$$

where c is any integer.

2. *For fixed c, the dihedral group action on numerically exceptional collections (see Sec. 3.1.7) has two orbits:*

$$I_c \to IV_c \to I_{-c} \to IV_{-c} \to I_c,$$
$$II_c \to III_c \to II_{-c} \to III_{-c} \to II_c.$$

As explained in Sec. 3.1.7, the Weyl group of $\mathbb{P}^1 \times \mathbb{P}^1$ acts on numerical exceptional collections on X, interchanging I_c with II_c and III_c with IV_c. The difference is that the Weyl group action does not lift to exceptional collections, so there are two orbits. Thus we need only consider numerically exceptional collections of line bundles of type I_c or II_c.

2.1 $(\mathbb{Z}/3)^2$-Beauville surface

ec!Z3-Beauville

This surface was discovered by Beauville and first described in [21], but for similar examples see also [8]. Let X be a $(\mathbb{Z}/3)^2$-cover of $Y = \mathbb{P}^1 \times \mathbb{P}^1$ branched over eight lines, four in each ruling. We label these branch divisors $\Delta_1, \ldots, \Delta_8$ (see Figure 5). Clearly, the branch locus has two free parameters, and in fact, the $(\mathbb{Z}/3)^2$-Beauville surfaces form a two dimensional irreducible connected component of the moduli space [8], so $h^1(T_X) = 2$ and $h^2(T_X) = 8$.

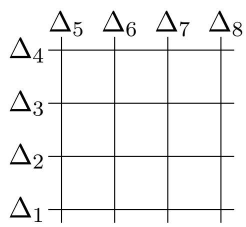

Figure 5: The branch locus for a Beauville surface with $G = (\mathbb{Z}/3)^2$ fig!Beauville-Z

The cover $\varphi\colon X \to Y$ and maximal abelian cover $\psi\colon A \to Y$ are determined by the maps $\Phi\colon H_1(Y - \Delta, \mathbb{Z}) \to (\mathbb{Z}/3)^2$ and $\Psi\colon H_1(Y - \Delta, \mathbb{Z}) \to (\mathbb{Z}/3)^6$ as shown in the table

	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6	Δ_7	Δ_8
$\Phi(D)$	g_1	g_2	$2g_1$	$2g_2$	$g_1 + g_2$	$g_1 + 2g_2$	$2g_1 + 2g_2$	$2g_1 + g_2$
$\Psi(D) - \Phi(D)$	0	0	g_3	$2g_3$	g_4	g_5	g_6	$2(g_4 + g_5 + g_6)$

The small quotient group $G \cong (\mathbb{Z}/3)^2$ is generated by g_1, g_2 and T is generated by $g_3, \ldots, g_6$.

Remark 2.1 The original construction [21] of X is to take the free $(\mathbb{Z}/3)^2$-quotient of a product $C_1 \times C_2$ of two special curves of genus 6. This realises a subgroup $(\mathbb{Z}/3)^2$ of the full torsion group $\operatorname{Tors} X = (\mathbb{Z}/3)^4$. Using this quotient construction, many exceptional collections of line bundles on X with numerical type I_1 were constructed and studied in [36]. We use abelian covers to completely enumerate all exceptional collections of line bundles on X, of any numerical type.

Let D_i denote the reduced pullback of Δ_i. Then the torsion free part of $\operatorname{Pic} X$ is based by D_1 and D_5, with intersection form $\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$.

!Z3-coordinates **Lemma 2.2** *The coordinates of each $\mathcal{O}_X(D_i)$ are*

$$\begin{aligned} \mathcal{O}_X(D_1) &= \mathcal{O}_X(1,0) & \mathcal{O}_X(D_5) &= \mathcal{O}_X(0,1) \\ \mathcal{O}_X(D_2) &= \mathcal{O}_X(1,0)[1,0,1,0] & \mathcal{O}_X(D_6) &= \mathcal{O}_X(0,1)[0,1,2,0] \\ \mathcal{O}_X(D_3) &= \mathcal{O}_X(1,0)[0,2,2,1] & \mathcal{O}_X(D_7) &= \mathcal{O}_X(0,1)[0,1,0,2] \\ \mathcal{O}_X(D_4) &= \mathcal{O}_X(1,0)[1,2,2,1] & \mathcal{O}_X(D_8) &= \mathcal{O}_X(0,1)[0,1,0,0] \end{aligned}$$

With this basis, using (8) we have

$$\mathcal{O}_X(K_X) = \mathcal{O}_X(2,2)[1,2,2,2]. \qquad (7)$$ eqn!Z3-canonica

2.1.1 The semigroup of effective divisors

In this section we prove:

lem!Z3-cone **Proposition 2.1** *The semigroup of effective divisors on X is generated by $D_1, \dots, D_8$.*

This should be compared with the results of [1] on primary Burniat surfaces and the discussion of Sec. 3.1.6.

We introduce some notation. Define $\mathfrak{E}$ to be the semigroup generated by $D_1, \dots, D_8$. It is convenient to consider $\mathfrak{E}$ as the image of the multiplicative semigroup M of monomials in the bigraded polynomial ring $\mathbb{Z}[x_1, x_2, x_3, x_4, y_1, y_2, y_3, y_4]$ under the homomorphism $x_i \mapsto D_i$, $y_i \mapsto D_{4+i}$. The x_i have bidegree $(1,0)$, and y_i have bidegree $(0,1)$. We abuse notation to consider monomials in M as elements of $\mathfrak{E}$ when appropriate. Let $t\colon \mathfrak{E} \to \operatorname{Tors} X$ be the semigroup homomorphism defined in Sec. 3.1.6, sending each D_i to its associated torsion twist according to Lem. 2.2.

Since K_X is ample, we have

lem!Z3-num-eff **Lemma 2.3** *If $\mathcal{O}(a,b)(\tau)$ is an effective line bundle on X, then $a \geq 0$ and $b \geq 0$.*

We analyse the possible values for a and b.

Lemma 2.4 *If $a, b \geq 2$, then $\mathcal{O}_X(a,b)(\tau)$ is effective for all τ in $\operatorname{Tors} X$, unless $a = b = 2$ and $\tau = [1,2,2,2]$.*

Proof Consider the set $M_{(2,2)}$ of monomials of bidegree $(2,2)$. We use the computer [20] to check that the image of $M_{(2,2)}$ under t is precisely $\operatorname{Tors} X - \{[1,2,2,2]\}$. Moreover, the missing torsion twist is that of K_X, which is not effective, because $p_g(X) = 0$.

On the other hand, we also check that $t(M_{(3,2)}) = t(M_{(2,3)}) = \operatorname{Tors} X$, so every line bundle of bidegree $(3,2)$ or $(2,3)$ is effective. Now for any $a \geq 3$, we see that $x_1^{a-3} y_1^{b-2} M_{(3,2)}$ gives a global section for each $\mathcal{O}_X(a,b)(\tau)$. A similar argument works for $b \geq 3$. $\square$

It remains to check what happens if $a \leq 1$ or $b \leq 1$. We suppose the latter (the case $a \leq 1$ is similar).

lem!blesseq1 **Lemma 2.5** *Suppose $b \leq 1$. The line bundle $\mathcal{O}_X(a,b)(\tau)$ is effective if and only if there is a monomial m in $M_{(a,b)}$ such that $t(m) = \tau$.*

Proof **Case $b = 0$.** For $a < 6$, we check effectivity of each line bundle directly. This is a finite number of line bundles, and so we use the computer [20]. Note that if $a < a'$, then $t(M_{(a,0)}) \subseteq t(M_{(a',0)})$. Moreover, for $a \geq 6$, $t(M_{(a,0)})$ stabilises to $H = \{[\alpha, \beta, \gamma, 2\beta] : \alpha, \beta, \gamma \in \mathbb{Z}/3\}$. Indeed, H is a subgroup of $\operatorname{Tors} X$, so it is closed under composition of torsion elements. Thus if $a \geq 6$ then $\mathcal{O}_X(a,0)(\tau)$ is effective for any τ in H.

Now fix $a \geq 6$ and τ in $\operatorname{Tors} X - H$. We show that $\mathcal{O}_X(a,0)(\tau)$ is not effective. Write $a = 6 + 3j + k$ where $j \geq 0$ and $0 \leq k \leq 2$. Then by Lemma 2.2,

$$\varphi_* \mathcal{O}_X(a,0)(\tau) = \varphi_* L(kD_1) \otimes \mathcal{O}_Y(j\Delta_1) = \varphi_* L(kD_1) \otimes \mathcal{O}_Y(j,0),$$

where $L = \mathcal{O}_X(6,0)(\tau)$. Thus if each summand of $\varphi_* L(kD_1)$ for $0 \leq k \leq 2$ has negative degree in the second factor, we see that $\mathcal{O}_X(a,0)(\tau)$ can not be effective for any $a \geq 6$. We have again reduced the problem to checking a finite number of line bundles, and this is done by computer in [20].

Case $b = 1$. The argument is similar to the previous case, so we give only a sketch. First check $a < 4$ directly. Then for $a \geq 4$, the image $t(M_{(a,1)})$ stabilises to $H \cup [0,1,0,0]H$, the union of two cosets of H in $\operatorname{Tors} X$. This can be seen directly from Lemma 2.2. The other torsion twists are ineffective for any $a \geq 4$, by a similar computation to that of case $b = 0$ above. $\square$

2.1.2 Acyclic line bundles

Now, by the Riemann–Roch theorem, the numerically acyclic line bundles on X are $\mathcal{O}(1,k)(\tau)$ and $\mathcal{O}(k,1)(\tau)$. Thus we may use Proposition 2.1 to find all acyclic line bundles.

prop!Z3-acyclic **Proposition 2.2** *For $k \geq 4$ or $k \leq -2$, the acyclic sets on X are*

$$\mathcal{A}(\mathcal{O}_X(1,k)) = \mathcal{S}, \quad \mathcal{A}(\mathcal{O}_X(k,1)) = \mathcal{T},$$

where

$$\mathcal{S} = \{[2, \alpha, \beta, \gamma] : \alpha, \beta, \gamma \in \mathbb{Z}/3\}, \ \mathcal{T} = \{[\alpha, \beta, \gamma, 2-\beta] : \alpha, \beta, \gamma \in \mathbb{Z}/3\}.$$

Proof We prove that $\mathcal{A}(\mathcal{O}(k,1)) = \mathcal{T}$ for $k \geq 4$. The acyclic set $\mathcal{A}(\mathcal{O}(1,k))$ for $k \geq 4$ can be calculated in the same way, and the negative cases follow by Serre duality.

Fix an integer $k \geq 4$. Then by Lem. 2.5, $\mathcal{O}_X(k,1)(\tau)$ is not effective if and only if τ is an element of $\operatorname{Tors} X - (H \cup [0,1,0,0]H) = \mathcal{T}$. Now by Serre duality and (7),

$$H^2(\mathcal{O}_X(k,1)[\alpha,\beta,\gamma,2-\beta]) = H^0(\mathcal{O}_X(2-k,1)([1-\alpha,2-\beta,2-\gamma,\beta]),$$

which also vanishes by Lem. 2.5, or if $k \geq 3$, we can use Lem. 2.3. Thus $\mathcal{A}(\mathcal{O}_X(1,k)) = \mathcal{T}$ for all $k \geq 4$. $\square$

In fact, the same proof shows that $\mathcal{S} \subset \mathcal{A}(\mathcal{O}_X(1,k))$ and $\mathcal{T} \subset \mathcal{A}(\mathcal{O}_X(k,1))$ for any integer k. For values of k between -1 and 3, there are a few extra acyclic twists, because the image of t has not yet stabilised to its maximum. These can be checked directly, using Prop. 2.1 as before.

L	$\mathcal{A}(L)$
$\mathcal{O}_X(1,-1)$	$\mathcal{S}, [1,1,1,0], [0,1,2,0]$
$\mathcal{O}_X(1,0)$	$\mathcal{S}, [0,2,0,0], [1,2,0,0], [1,1,1,0], [0,2,1,0], [1,2,1,0], [0,1,2,0], [0,2,2,0], [1,2,2,0], [1,0,1,1], [0,0,2,1], [1,2,1,2], [0,2,2,2]$
$\mathcal{O}_X(1,1)$	$\mathcal{S} \cup \mathcal{T}, [1,0,0,0], [0,0,1,0], [1,2,1,2], [0,2,2,2]$
$\mathcal{O}_X(1,2)$	$\mathcal{S}, [1,0,0,0], [0,0,1,0], [1,2,0,1], [0,2,1,1], [0,0,0,2], [1,0,0,2], [1,1,0,2], [0,0,1,2], [1,0,1,2], [0,1,1,2], [0,0,2,2], [1,0,2,2]$
$\mathcal{O}_X(1,3)$	$\mathcal{S}, [1,1,0,2], [0,1,1,2]$

In the other direction,

L	$\mathcal{A}(L)$
$\mathcal{O}_X(-1,1)$	$\mathcal{T}, [1,0,2,1], [1,1,1,2]$
$\mathcal{O}_X(0,1)$	$\mathcal{T}, [1,0,0,0], [2,0,0,0], [1,1,0,0], [2,1,2,0], [2,0,0,1], [2,2,1,1], [1,0,2,1], [1,2,2,1], [1,1,1,2], [1,2,1,2], [2,2,1,2], [2,1,2,2]$
$\mathcal{O}_X(2,1)$	$\mathcal{T}, [2,1,0,0], [0,0,1,0], [2,0,1,0], [0,1,1,0], [0,0,0,1], [0,2,0,1], [2,0,1,1], [2,2,2,1], [2,1,0,2], [0,1,2,2], [0,2,2,2], [2,2,2,2]$
$\mathcal{O}_X(3,1)$	$\mathcal{T}, [0,1,1,0], [0,2,0,1]$

Many exceptional collections on X of numerical type I_1 and with formal A_∞-algebra were constructed in [36]. We can classify all exceptional collections of line bundles on X, of any numerical type. The enumeration is summarised below, but see [20] for details.

Proposition 2.3 *Exceptional collections of line bundles on the $\mathbb{Z}/3$-Beauville surface are enumerated in the table below. The integer $c \geq 0$ determines the numerical type of the exceptional collection, either I_c or II_c. The number of type I_c is equal to the number of type II_c.*

c	0	1	2	≥ 3
#(*Exceptional collections*)	6661	3613	2213	$2187 = 3^7$

We display a sample exceptional collection of type I_1

$$\mathcal{O}_X,\ \mathcal{O}_X(-1,0)[0,1,0,0],\ \mathcal{O}_X(0,-1)[2,2,0,0],\ \mathcal{O}_X(-1,-1)[1,0,1,0]$$

	0	1	2	3
0	1	q^2	q^2	$4q^2$
1	1	0	0	$6q^2$
2	1	$2q^2$	$6q^2$	$8q^2$
3	1	$4q^2$	$6q^2$	$6q^2$

Table 7: Ext-table of an exceptional collection on the $(\mathbb{Z}/3)^2$-Beauville surface tab!ext-Beauvil

Table 7 is the Ext-table of this exceptional collection. We see that there are no nonzero Ext^1-groups. Hence the A_∞-algebra is formal, and the height is 4. Thus the Hochschild cohomology of the corresponding quasiphantom category is $HH^0(\mathcal{A}) = \mathbb{C}$, $HH^1(\mathcal{A}) = 0$, $HH^2(\mathcal{A}) = \mathbb{C}^2$, $HH^3(\mathcal{A}) \supset \mathbb{C}^8$.

2.2 $(\mathbb{Z}/5)^2$-Beauville surface

ec!Z5-Beauville

We consider the $(\mathbb{Z}/5)^2$-Beauville surface, which was first described in [11] and [21]. Exceptional collections of line bundles on this surface were classified by Galkin and Shinder [24], which was a major influence on our overall approach. We recover the results of [24] as a test case for our methods.

This time X is a $(\mathbb{Z}/5)^2$-cover of $Y = \mathbb{P}^1 \times \mathbb{P}^1$ branched over six lines, three in each ruling. This branch configuration is rigid, and in fact the moduli space of such Beauville surfaces is zero dimensional and smooth. The torsion group of X is $\mathrm{Tors}\, X \cong (\mathbb{Z}/5)^2$, which is fully realised by the standard construction of X as a free $(\mathbb{Z}/5)^2$-quotient of $C_1 \times C_2$, where C_i are Fermat quintic curves. Thus $C_1 \times C_2$ is the maximal abelian cover A (this description of A is not necessary for our approach).

The maps $\Phi\colon H_1(Y-\Delta,\mathbb{Z}) \to (\mathbb{Z}/5)^2$ and $\Psi\colon H_1(Y-B,\mathbb{Z}) \to \widetilde{G} \cong (\mathbb{Z}/5)^4$ determining the covers are defined in the following table

	Δ_1	Δ_2	Δ_3	Δ_4	Δ_5	Δ_6
$\Phi(D)$	g_1	g_2	$4g_1+4g_2$	g_1+2g_2	$3g_1+4g_2$	g_1+4g_2
$\Psi(D)-\Phi(D)$	0	0	0	g_3	g_4	$4g_3+4g_4$

The reduced pullbacks D_1 (respectively D_4) of Δ_1 (resp. Δ_4) are a basis for the free part of $\operatorname{Pic} X$. As usual, the other reduced pullbacks may be written in terms of this basis, and we have

!Z5-coordinates **Lemma 2.6**

$$\begin{aligned} \mathcal{O}_X(D_1) &= \mathcal{O}_X(1,0), & \mathcal{O}_X(D_5) &= \mathcal{O}_X(0,1), \\ \mathcal{O}_X(D_2) &= \mathcal{O}_X(1,0)[1,1], & \mathcal{O}_X(D_5) &= \mathcal{O}_X(0,1)[1,4], \\ \mathcal{O}_X(D_3) &= \mathcal{O}_X(1,0)[4,2], & \mathcal{O}_X(D_6) &= \mathcal{O}_X(0,1)[1,0]. \end{aligned}$$

By (8), we have

$$\mathcal{O}_X(K_X) = \mathcal{O}_X(2,2)[3,3].$$

em!Z5-effective **Lemma 2.7** *The semigroup $\mathfrak{E}$ of effective divisors on X is the set of positive integer linear combinations of $D_1,\ldots,D_6$.*

The proof of this Lemma is similar to that of Prop. 2.1.

As in Sec. 3.1.6, we define a semigroup homomorphism $t\colon \mathfrak{E} \to \operatorname{Tors} X$ using the torsion twists from Lem. 2.6. Using Lem. 2.7, we list all acyclic line bundles on X in the following table. We note that the restrictions $t|_{\mathfrak{E}_{(1,j)}}$ and $t|_{\mathfrak{E}_{(k,1)}}$ are surjective for $j \geq 5$ and $k \geq 4$.

L	$\mathcal{A}(L)$
$\mathcal{O}_X(1,-2)$	$[2,0]$
$\mathcal{O}_X(1,-1)$	$[2,0], [3,0], [3,4]$
$\mathcal{O}_X(1,0)$	$[2,0], [3,0], [4,0], [0,1], [4,3], [3,4], [4,4]$
$\mathcal{O}_X(1,1)$	$[3,0], [4,0], [0,3], [4,3], [0,4], [3,4], [4,4]$
$\mathcal{O}_X(1,2)$	$[4,0], [3,2], [0,3], [1,3], [4,3], [0,4], [4,4]$
$\mathcal{O}_X(1,3)$	$[0,3], [1,3], [0,4]$
$\mathcal{O}_X(1,4)$	$[1,3]$
$\mathcal{O}_X(-1,1)$	$[4,2]$
$\mathcal{O}_X(0,1)$	$[4,2], [0,3], [4,3], [3,4]$
$\mathcal{O}_X(2,1)$	$[3,0], [4,0], [4,1], [0,4]$
$\mathcal{O}_X(3,1)$	$[4,1]$

Up to choices of coordinates, these are precisely the acyclic line bundles listed in [24], and there are no others. It seems that the rigidity of X is reflected in the small number of acyclic line bundles.

Using this list of acyclic line bundles, and Lemma 2.1, we can classify all exceptional collections of line bundles of length four on X. Here is the complete list, which form two orbits, replicating results of [24].

$$\begin{cases} I_{-1} & \mathcal{O}, \quad \mathcal{O}(-1,0)[0,4], & \mathcal{O}(-2,-1)[1,0], & \mathcal{O}(-3,-1)[1,4] \\ IV_{-1} & \mathcal{O}, \quad \mathcal{O}(-1,-1)[1,1], & \mathcal{O}(-2,-1)[1,0], & \mathcal{O}(-1,-2)[2,3] \\ I_1 & \mathcal{O}, \quad \mathcal{O}(-1,0)[0,4], & \mathcal{O}(0,-1)[1,2], & \mathcal{O}(-1,-1)[1,1] \\ IV_1 & \mathcal{O}, \quad \mathcal{O}(1,-1)[1,3], & \mathcal{O}(0,-1)[1,2], & \mathcal{O}(-1-2)[2,3] \end{cases}$$

$$\begin{cases} II_0 & \mathcal{O}, \quad \mathcal{O}(0,-1)[1,2], & \mathcal{O}(-1,-1)[1,1], & \mathcal{O}(-1,-2)[2,3] \\ III_0 & \mathcal{O}, \quad \mathcal{O}(-1,0)[0,4], & \mathcal{O}(-1,-1)[1,1], & \mathcal{O}(-2,-1)[1,0] \end{cases}$$

We do not continue the analysis of quasi-phantoms, since it appears in [24]. We only verify that our results are consistent.

3 Keum–Naie surface with $K^2 = 4$

sec!Keum-Naie

In this section we investigate a construction of Keum–Naie surfaces, which just fails to satisfy our assumptions from Sec. 3.1. The problem is that the maximal abelian cover $A \to X$ does not factor through a Galois cover of the del Pezzo surface Y. Thus while we can describe the free part of $\operatorname{Pic} X$ in terms of reduced pullbacks of branch divisors, we can only describe an index 2 subgroup of $\operatorname{Tors} X$ using our approach. We have used various numerical exceptional collections to search for exceptional collections of maximal length on X, but without success.

3.1 Construction and basic properties of the surface

Keum–Naie surfaces were discovered independently in [38] and [30] as branched double covers of Enriques surfaces with eight nodes. The connected component of the moduli space containing Keum–Naie surfaces has dimension 6, and the torsion group is $(\mathbb{Z}/2)^3 \times \mathbb{Z}/4$.

Following [10] and [3], we consider a special 2-dimensional subfamily of Keum–Naie surfaces. Each surface X in the subfamily admits a singular $\mathbb{Z}/2 \times \mathbb{Z}/4$-cover of $\mathbb{P}^1 \times \mathbb{P}^1$ branched over eight lines, four in each ruling. The branch configuration is shown in Figure 6.

The map $\Phi \colon H_1(\mathbb{P}^1 \times \mathbb{P}^1 - \Delta, \mathbb{Z}) \to \mathbb{Z}/2 \times \mathbb{Z}/4$ governing the cover $X \to \mathbb{P}^1 \times \mathbb{P}^1$ is

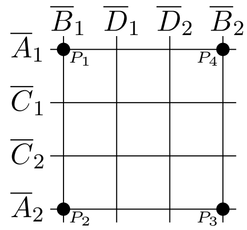

Figure 6: The Keum–Naie configuration with $K^2 = 4$ fig!Keum-Naie

described in the table below.

Γ	$\overline{A}_1$	$\overline{A}_2$	$\overline{B}_1$	$\overline{B}_2$	$\overline{C}_1$	$\overline{C}_2$	$\overline{D}_1$	$\overline{D}_2$
$\Phi(\Gamma)$	h	$3h$	$g_1 + h$	$g_1 + 3h$	g_1	g_1	$g_1 + 2h$	$g_1 + 2h$
$\Psi'(\Gamma) - \Phi(\Gamma)$	0	g_2	g_3	g_4	0	g_2	g_5	$g_3 + g_4 + g_5$

In the table, g_i have order 2 while h has order 4. We see that the four divisors $\overline{A}_i$, $\overline{B}_i$ have inertia group $\mathbb{Z}/4$ under φ, while the other branch divisors have inertia group $\mathbb{Z}/2$. Moreover, the four points $P_1, \dots, P_4$ correspond to $\frac{1}{2}(1,1)$ singularities on X. We blow up the P_i to get a nonsingular cover of a weak del Pezzo surface Y of degree 4. Write $\overline{E}_i$ for the (-1)-curve corresponding to P_i. We use the same labels for the strict transforms under the blow up, so $\overline{A}_i$, $\overline{B}_i$ are (-2)-curves on Y. Note that by formula (1) the curves $\overline{E}_i$ have inertia group $\mathbb{Z}/2$.

The following proposition explains why we can not find exceptional collections on the Keum–Naie surface.

Proposition 3.1 *Let A be the maximal abelian cover of X. The composite map $A \to X \to Y$ is not Galois.*

Proof The torsion group of X is $(\mathbb{Z}/2)^3 \times \mathbb{Z}/4$, so if $\psi\colon A \to Y$ is Galois, we have a surjective homomorphism $\Psi\colon H_1(Y - \Delta, \mathbb{Z}) \to (\mathbb{Z}/2)^4 \times (\mathbb{Z}/4)^2$. Now consider $\Psi(\overline{E}_1) = \Psi(\overline{A}_1 + \overline{B}_1)$. The order of $\Psi(\overline{E}_i)$ must be 2, while the orders of $\Psi(\overline{A}_1)$ and $\Psi(\overline{B}_1)$ must be 4. We have similar requirements coming from the other $\overline{E}_i$. There is no surjective homomorphism Ψ satisfying these conditions. $\square$

We define A' to be the intermediate Galois cover $A' \to X$ corresponding to the subgroup $(\mathbb{Z}/2)^4$ of index 2 in $\operatorname{Tors} X$, generated by $g_2, \dots, g_5$. The composite map $A' \to X \to Y$ is Galois, with defining map $\Psi'\colon H_1(Y - \Delta, \mathbb{Z}) \to \mathbb{Z}/4 \times (\mathbb{Z}/2)^5$ in the table above.

3.2 The Picard group

lem!KN-picard **Lemma 3.1** *The reduced pullbacks $e_0 = A_1 + B_1 + E_1$, $e_1 = A_2 + B_1 + E_2$, $e_2 = A_1$, $e_3 = A_2$, $e_4 = B_1$, $e_5 = B_2$ generate the Picard lattice of X with intersection matrix*

$U \oplus \operatorname{diag}(-1,-1,-1,-1)$, *where* $U = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Proof We first show that linear combinations of the quoted divisors generate $\operatorname{Pic} Y$. We use the following linear equivalences on Y

$$\begin{aligned} \overline{C}_1 \sim \overline{C}_2 \sim \overline{A}_1 + \overline{E}_1 + \overline{E}_4 \sim \overline{A}_2 + \overline{E}_2 + \overline{E}_3 \\ \overline{D}_1 \sim \overline{D}_2 \sim \overline{B}_1 + \overline{E}_1 + \overline{E}_2 \sim \overline{B}_2 + \overline{E}_3 + \overline{E}_4, \end{aligned} \tag{8}$$

eqn!KN-lin-equi

to express E_3 and E_4 in terms of the basis. The rest of the proof is similar to that of Lemma 3.3. Checking the intersection matrix requires some care with the definition of reduced pullback, because the inertia groups of φ are not uniform. For example, we actually have $4e_0 = \varphi^*(\overline{A}_1 + 2\overline{E}_1 + \overline{B}_1)$ and $4e_1 = \varphi^*(\overline{A}_2 + 2\overline{E}_2 + \overline{B}_1)$, so that $16 e_0 \cdot e_1 = \deg(\varphi)\cdot 2$, and hence $e_0 \cdot e_1 = 1$. $\square$

Remark 3.1 By Lemma 3.1, we see that A_1 is an elliptic curve of self-intersection -1 on X, even though $\overline{A}_1$ is a (-2)-curve on Y. In other words, assumptions (A1) and (A2) hold for the Keum–Naie surface, but (A3) does not. Instead we get an isometry from the abstract lattice $\mathbb{Z}^{1,5} \to \operatorname{Pic} X/\operatorname{Tors} X$, under which the image of $2e_0 + 2e_1 - \sum_{i=2}^{5} e_i$ is the class of $\mathcal{O}_X(K_X)$ modulo torsion.

We compute the coordinates of the reduced pullback of each irreducible branch component using the basis provided by Lemma 3.1.

Lemma 3.2 *We have*

$$\begin{aligned}
&\mathcal{O}_X(A_1) = \mathcal{O}_X(0,0,1,0,0,0) && \mathcal{O}_X(D_1) = \mathcal{O}_X(1,1,-1,-1,0,0)[0,1,0,0] \\
&\mathcal{O}_X(A_2) = \mathcal{O}_X(0,0,0,1,0,0) && \mathcal{O}_X(D_2) = \mathcal{O}_X(1,1,-1,-1,0,0)[0,1,0,1] \\
&\mathcal{O}_X(B_1) = \mathcal{O}_X(0,0,0,0,1,0) && \mathcal{O}_X(E_1) = \mathcal{O}_X(1,0,-1,0,-1,0) \\
&\mathcal{O}_X(B_2) = \mathcal{O}_X(0,0,0,0,0,1) && \mathcal{O}_X(E_2) = \mathcal{O}_X(0,1,0,-1,-1,0) \\
&\mathcal{O}_X(C_1) = \mathcal{O}_X(1,1,0,0,-1,-1)[1,1,1,0] && \mathcal{O}_X(E_3) = \mathcal{O}_X(1,0,0,-1,0,-1)[0,0,0,1] \\
&\mathcal{O}_X(C_2) = \mathcal{O}_X(1,1,0,0,-1,-1)[1,0,0,1] && \mathcal{O}_X(E_4) = \mathcal{O}_X(0,1,-1,0,0,-1)[0,1,1,1]
\end{aligned}$$

Proof This is similar to Lemma 3.3. One minor point, in computing the multidegrees. The linear equivalences (8) on Y pull back to X giving numerical equivalences

$$\begin{aligned} C_1 \equiv C_2 \equiv 2A_1 + E_1 + E_4 \equiv 2A_2 + E_2 + E_3 \\ D_1 \equiv D_2 \equiv 2B_1 + E_1 + E_2 \equiv 2B_2 + E_3 + E_4. \end{aligned}$$

These can be rearranged to give

$$A_1 + B_1 + E_1 \equiv A_2 + B_2 + E_3, \ A_2 + B_1 + E_2 \equiv A_1 + B_2 + E_4,$$

which is used to express each reduced pullback in terms of the basis from Lemma 3.1. $\square$

Lemma 3.3 *By formula* (8) *and Lem. 3.1,* $\mathcal{O}_X(K_X) = \mathcal{O}_X(2, 2, -1, -1, -1, -1)$. $\square$

We conclude by noting that we have searched for, but not found any exceptional collections of maximal length on the Keum–Naie surface. It seems that our subgroup of $\operatorname{Tors} X$ is too small to allow us the freedom to find any. On a related note, exceptional collections of maximal length have not been discovered on the Burniat–Campedelli surface with $K^2 = 2$ (see [1]), and some Beauville surfaces considered in [36]. Here the situation is more straightforward, because these surfaces fail to satisfy assumption (A1) and (A2) respectively.